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ON THE UNIVALENCE OF A CERTAIN INTEGRAL

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1. **Introduction.** Let S be the class of functions $f(z)$ regular, univalent in $|z|<1$ and normalized by $f(0)=0, f'(0)=1$. On the other hand, let S^* and K be the subclass of S starlike and convex functions respectively.

It is well known that a function $f(z) \in S$ belongs to S^* if and only if

$$\operatorname{Re}(zf'(z)/f(z)) > 0 \quad \text{in } |z| < 1$$

and a function $f(z) \in S$ belongs to K if and only if

$$1+\operatorname{Re}(zf''(z)/f'(z)) > 0 \quad \text{in } |z| < 1.$$

In the recent papers [2], [3], [9], [11], for the univalence of the functions

$$g(z) = \int_0^z \left(\frac{f(t)}{t}\right)^\alpha dt \quad \text{and} \quad g(z) = \int_0^z (f'(t))^\alpha\, dt$$

was studied.

For instance, the following theorems are obtained in [2], [9], [11].

THEOREM A. *If $f(z)$ belongs to S and is close-to-convex, then*

$$g(z) = \int_0^z (f'(t))^\alpha\, dt$$

belongs to S for $0 \leqq \alpha \leqq 1$.

THEOREM B. *Suppose $f(z) \in S$ is close-to-convex. Then*

$$g(z) = \int_0^z \left(\frac{f(t)}{t}\right)^\alpha dt$$

belongs to S for $0 \leqq \alpha \leqq 1$.

THEOREM C. *Let $f(z) \in S$ and*

$$g(z) = \int_0^z \left(\frac{f(t)}{t}\right)^\alpha dt.$$

Then $g(z) \in S$ for $0 \leqq \alpha \leqq ((1025)^{1/2}-25)/100$.

In this paper we improve Theorem C and others.

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2. The main theorems.

LEMMA 1. *Let* $f(z)=z+a_2z^2+\cdots$ *be regular in* $|z|<1$. *If* $f(z)$ *satisfies*

$$1+\text{Re}\,\frac{zf''(z)}{f'(z)} > -\frac{1}{2} \quad in\ |z| < 1,$$

then $f(z)$ *is univalent in* $|z|<1$.

We owe this lemma to Ozaki [10], [13].

THEOREM 1. *Let*

$$f(z) = z+\sum_{n=2}^{\infty} a_n z^n \in S^*$$

and

$$g(z) = \int_0^z \left(\frac{f(t)}{t}\right)^{\alpha} dt.$$

Then $g(z) \notin S$ *for* $0\leqq\alpha\leqq 1.5$ *but for* $\alpha_0<\alpha$, *there exists a function* $f(z)\in S^*$ *such that* $g(z)\in S$ *where* α_0 *is the smallest positive root of the equation*

$$\alpha(2\alpha+1)(\alpha+1)-24 = 0.$$

Proof. It follows that

$$1+zg''(z)/g'(z) = 1+\alpha(zf'(z)/f(z)-1).$$

Letting $0<\alpha\leqq 1.5$ we have

$$1+\text{Re}\,\frac{zg''(z)}{g'(z)} = 1-\alpha+\alpha\,\text{Re}\,\frac{zf'(z)}{f(z)} > 1-\alpha \geqq -\frac{1}{2}.$$

Therefore we have that $g(z)\in S$ for $0\leqq\alpha\leqq 1.5$. On the other hand, if we let $f(z)=z/(1-z)^2\in S^*$ and $g(z)\in S$, then we must have from [4, p. 2] and [5, p. 134]

$$g'(z) = \frac{1}{(1-z)^{2\alpha}} = 1+2\alpha z+\frac{2\alpha(2\alpha+1)}{2!}z^2+\frac{2\alpha(2\alpha+1)(2\alpha+2)}{3!}z^3+\cdots$$

and therefore

$$(1) \qquad |2\alpha| \leqq 2^2, \qquad \left|\frac{2\alpha(2\alpha+1)}{2!}\right| \leqq 3^2$$

and

$$\left|\frac{2\alpha(2\alpha+1)(2\alpha+2)}{3!}\right| \leqq 4^2.$$

Letting α_0 be a positive real number, we must have the following inequality from (1):

$$0 < \alpha \leqq \alpha_0 < \frac{(73)^{1/2}-1}{4} < 2$$

where α_0 is the smallest positive root of the equation

$$\alpha(2\alpha+1)(\alpha+1)-24 = 0.$$

This completes our proof.

THEOREM 2. *Let*

$$f(z) = z+\sum_{n=2}^{\infty} a_n z^n \in K$$

and

$$g(z) = \int_0^z \left(\frac{f(t)}{t}\right)^\alpha dt.$$

Then $g(z) \in S$ for $0 \leqq \alpha \leqq 3$ but for $\alpha_1 < \alpha$, there exists a function $f(z) \in K$ such that $g(z) \notin S$ where α_1 is the smallest positive root of the equation $\alpha(\alpha+1)(\alpha+2)-96=0$.

Proof. It is well known [6], [12] that

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \frac{1}{2} \quad \text{in } |z| < 1.$$

Applying the same method as in the proof of Theorem 1 we have

$$1+\operatorname{Re} \frac{zg''(z)}{g'(z)} = 1-\alpha+\operatorname{Re} \alpha \frac{zf'(z)}{f(z)} > 1-\alpha+\frac{1}{2}\alpha \geqq -\frac{1}{2}$$

if $0<\alpha \leqq 3$. Therefore $g(z) \in S$ for $0 \leqq \alpha \leqq 3$.

Putting $f(z)=z/(1-z) \in K$ and $g(z) \in S$, then we have

$$g'(z) = \frac{1}{(1-z)^\alpha} = 1+\alpha z+\frac{\alpha(\alpha+1)}{2!} z^2+\frac{\alpha(\alpha+1)(\alpha+2)}{3!} z^3+\cdots$$

and therefore we have also as in the proof of Theorem 1

$$(2) \qquad |\alpha| \leqq 2^2, \qquad \left|\frac{\alpha(\alpha+1)}{2!}\right| \leqq 3^2$$

and

$$\left|\frac{\alpha(\alpha+1)(\alpha+2)}{3!}\right| \leqq 4^2.$$

Letting α be a positive real number, we must have from (2) the following

$$0 < \alpha \leqq \alpha_1 < ((72)^{1/2}-1)/2 < 4$$

where α_1 is the smallest positive root of the equation

$$\alpha(\alpha+1)(\alpha+2)-96 = 0.$$

This completes our proof and Theorem 2 is a stronger result than [9, Theorem 4].

THEOREM 3. *Let*

$$f(z) = z+\sum_{n=2}^{\infty} a_n z^n \in K$$

and

$$g(z) = \int_0^z (f'(t))^\alpha \, dt.$$

Then $g(z) \in S$ *for* $0 \leqq \alpha \leqq 1.5$ *but for* $\alpha_0 < \alpha$, *there exists a function* $f(z) \in K$ *such that* $g(z) \notin S$ *where* α_0 *is the smallest positive root of the equation*

$$\alpha(2\alpha+1)(\alpha+1)-24 = 0.$$

Proof. We have

$$1+\frac{zg''(z)}{g'(z)} = 1+\alpha\frac{zf''(z)}{f'(z)}$$

and so

$$1+\operatorname{Re}\frac{zg''(z)}{g'(z)} = 1-\alpha+\operatorname{Re}\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)$$
$$> 1-\alpha \geqq -\frac{1}{2}$$

if $0<\alpha\leqq 1.5$.

Therefore $g(z) \in S$ if $0 \leqq \alpha \leqq 1.5$ and this is a stronger result than [9, Theorem 3].

Putting $f(z) = z/(1-z) \in K$ and $g(z) \in S$, then we have $g'(z) = 1/(1-z)^{2\alpha}$.

By the same reason as in the proof of Theorem 1 we can complete our proof.

THEOREM 4. *Let*

$$f(z) = z+\sum_{n=2}^{\infty} a_n z^n$$

be regular in $|z|<1$, $\operatorname{Re} f'(z)>0$ *and*

$$g(z) = \int_0^z (f'(t))^\alpha \, dt.$$

Then $g(z) \in S$ *for* $-1 \leqq \alpha \leqq 1$.

Proof. It follows that

$$\operatorname{Re} g'(z) = \operatorname{Re}(f'(z))^\alpha > 0 \quad \text{in } |z| < 1$$

if $-1 \leqq \alpha \leqq 1$.

By Noshiro [8] we have $g(z) \in S$ for $-1 \leqq \alpha \leqq 1$.

LEMMA 2. *Let*

$$f(z) = z+\sum_{n=2}^{\infty} a_n z^n$$

be regular in $|z|<1$ *and*

$$|\{f, z\}| < \frac{2}{(1-r^2)^2}$$

for all z, $|z| = r < 1$, *where*

$$\{f, z\} = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2$$

is the Schwarzian derivative. Then $f(z)$ *is univalent in* $|z| < 1$.

The proof of this lemma can be found in [7].

LEMMA 3. *If* $f(z)$ *is regular in* $|z| < 1$, $f(0) = 0$ *and satisfying* $|f(z)| < 1$ *there, then*

$$|f'(z)| < 1 \quad \textit{or} \quad |f'(z)| < \frac{(1+|z|^2)^2}{4|z|(1-|z|^2)}$$

according as

$$|z| < \sqrt{2}-1 \quad \textit{or} \quad \sqrt{2}-1 \leqq |z| < 1.$$

These bounds are sharp.

A proof of this lemma can be found in [1].

THEOREM 5. *Let*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S$$

and

$$g(z) = \int_0^z \left(\frac{f(t)}{t}\right)^{\alpha} dt.$$

Then $g(z) \in S$ *for* $0 \leqq \alpha \leqq \alpha_2$ *where* α_2 *is the smallest positive root of the simultaneous equations* (5) *and*

$$\frac{(18425)^{1/2}-75}{800} < \alpha_2 < \frac{(24841)^{1/2}-125}{384}.$$

Proof. Let α be a positive real number and

$$F(z) = g''(z)/g'(z).$$

Then $F(z)$ is regular in $|z| < 1$ and we have also from [9, p. 396]

$$|F(z)| < 8\alpha \quad \text{in } |z| < \tfrac{1}{2}.$$

Let

$$G(z) = \{F(z/2) - F(0)\}/10\alpha \quad \text{in } |z| < 1.$$

Applying Lemma 3 and the same method as in the proof of [9] we have

$$|G'(z)| = \frac{1}{20\alpha}\left|F'\left(\frac{z}{2}\right)\right| \leqq \frac{(1+\rho^2)^2}{4\rho(1-\rho^2)} \quad \text{in } \sqrt{2}-1 \leqq |z| \leqq \rho < 1.$$

From the maximum principle we have

$$|F'(z)| \leqq \frac{5\alpha(1+\rho^2)^2}{\rho(1-\rho^2)} \quad \text{in } |z| \leqq \frac{\rho}{2}.$$

Hence we get

$$
\begin{aligned}
|\{g, z\}| &\leqq \left|\left(\frac{g''(z)}{g'(z)}\right)'\right| + \frac{1}{2}\left|\left(\frac{g''(z)}{g'(z)}\right)\right|^2 \\
&= |F'(z)| + \frac{1}{2}|F(z)|^2 \leqq \left\{32\alpha^2 + \frac{5\alpha(1+\rho^2)^2}{\rho(1-\rho^2)}\right\} \Big/ (1-r^2)^2 \quad \text{in } |z| = r \leqq \frac{\rho}{2}.
\end{aligned} \tag{3}
$$

In $\rho/2 \leqq |z| = r < 1$ we have from [9, p. 397]

$$
\begin{aligned}
|F'(z)| &\leqq \frac{2\alpha}{r(1-r)(1-\sqrt{r})} \\
&= \frac{2\alpha(1+r)^2(1+\sqrt{r})}{r(1-r^2)^2} \\
&< \frac{2\alpha}{(1-r^2)^2}\cdot\left\{2\left(1+\frac{\rho}{2}\right)^2\left(1+\left(\frac{\rho}{2}\right)^{1/2}\right)\rho^{-1}\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
|F(z)| &\leqq \frac{2\alpha}{r(1-r)} \\
&= \frac{2\alpha(1+r)}{r(1-r^2)} \\
&< \frac{2\alpha}{(1-r^2)}\left(1+\frac{2}{\rho}\right).
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
|\{g, z\}| &\leqq \frac{2\alpha}{(1-r^2)^2}\{2(1+\rho/2)^2(1+(\rho/2)^{1/2})\rho^{-1}\} + \frac{4\alpha^2}{(1-r^2)^2}(1+2/\rho)^2 \\
&= \{4\alpha^2(1+2/\rho)^2 + 4\alpha(1+\rho/2)^2(1+(\rho/2)^{1/2})\rho^{-1}\}/(1-r^2)^2 \\
&\qquad\qquad \text{in } \rho/2 \leqq |z| < 1.
\end{aligned} \tag{4}
$$

Putting

$$
\begin{aligned}
32\alpha^2 + \frac{5\alpha(1+\rho^2)^2}{\rho(1-\rho^2)} &= 4\alpha^2(1+2/\rho)^2 + 4\alpha(1+\rho/2)^2(1+(\rho/2)^{1/2})\rho^{-1} \\
&= 2
\end{aligned} \tag{5}
$$

and

$$
\sqrt{2} - 1 \leqq \rho < 1.
$$

Let α_2 be the smallest positive root of the simultaneous equations (5). Then we have

$$
|\{g, z\}| \leqq \frac{2}{(1-r^2)^2} \quad \text{in } |z| = r < 1
$$

if $0 < \alpha < \alpha_2$.

From Lemma 2 we have $g(z) \in S$ for $0 \leqq \alpha \leqq \alpha_2$. (For the case $\alpha = 0$, Theorem 5 is trivial.)

It can be verified that

$$\frac{(18425)^{1/2}-75}{800} < \alpha_2 < \frac{(24841)^{1/2}-125}{384}.$$

REMARK. If we put $\rho=\frac{1}{2}$ in (3) and (4) we have

$$|\{g, z\}| \leqq \frac{2}{(1-r^2)^2} \quad \text{in } |z| = r < \frac{1}{4}$$

if

$$0 \leqq \alpha \leqq \frac{(24841)^{1/2}-125}{384}$$

and

$$|\{g, z\}| \leqq \frac{2}{(1-r^2)^2} \quad \text{in } \tfrac{1}{4} \leqq |z| = r < 1$$

if

$$0 \leqq \alpha \leqq \frac{(18425)^{1/2}-75}{800}.$$

Therefore we have $g(z) \in S$ for at least

$$0 \leqq \alpha \leqq \frac{(18425)^{1/2}-75}{800}.$$

This is an improvement of [9, Theorem 1].

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